Takens' Theorem

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All of mathematical physics either concerns infinitesimal descriptions of reality or global ones[5]. A simple example of the former is a differential equation, whereas fields like gravitational fields and electromagnetic fields exemplify the latter. Differential Geometry and Differential Topology provide a language that unifies this description in the language of bundles. For instance, fields are defined as sections of appropriate bundles. In fact, a field theory, which is usually formalised as a variational calculus problem and its leading differential equation, can be prescribed in terms of special bundles called jet bundles. This formalism for field theory spans both classical and quantum field theory, after suitable modifications.

In the Lagrangian description, the core, unifying idea is the principle of least action. This is computed from a quantity called the **Lagrangian Density**, from which follow symmetry laws via Noether's Theorem[3]. In this article, we limit our focus to the former.

Mathematical Background

Let $\mathcal{C}$ be a category, $E, M \in Obj(\mathcal{C})$ and $\pi \in \mathtt{Hom}_{\mathcal{C}}(E, M)$. The pair (E, M, π) is called a **bundle** where E is called the **total space**, B is the **base space** of the bundle and π is called the **projection** of the bundle. A morphisms of two bundles $\pi_1 : E_1 \longrightarrow M_1$ and $\pi_2 : E_2 \longrightarrow M_2$ is given by the expected commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

In the category $\mathcal{C}$ of topological spaces, a particular case of a bundle is the **fiber bundle**: this is a tuple (E, M, π, F) where $\pi \in \mathtt{Hom}_{\mathcal{C}}(E, M)$ is surjective that is **locally trivial**: for each $x \in M$, there exists an open set $U_x \subset M$ of x such that $\varphi : \pi^{-1}(U_x) \xrightarrow{\sim} U_x \times F$ with φ compatible with the natural projection onto U_x. That is, the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U_x) & \xrightarrow{\varphi} & U_x \times F \\ {\scriptstyle \pi}\big\downarrow & \swarrow & \\ U_x & & \end{array}$$

It is in this sense that the space E looks locally like a product. The space F is called the **fiber** of the fiber bundle and U is called the **trivialising neighborhood**. A trivial example is given by $E = M \times F$ with π defined as its canonical projection and $\varphi = id$.

Again, a morphism between two fiber bundles (E_1, M_1, π_1, F_1) and (E_2, M_2, π_2, F_2) with trivializations $\varphi_1 : \pi_1^{-1}(U_1) \xrightarrow{\sim} U_1 \times F_1$ and $\varphi_2 : \pi_2^{-1}(U_2) \xrightarrow{\sim} U_2 \times F_1$, is given by the first commutative diagram with the additional constraint

$$\begin{array}{ccccccc} U_2 \times F_2 & \longleftarrow & \pi_2^{-1}(U_2) & \xleftarrow{\tilde{f}} & \pi_1^{-1}(U_1) & \xrightarrow{\varphi_1} & U_1 \times F_1 \\ & \searrow & \big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & \swarrow & \\ & & U_2 & \xleftarrow[f]{} & U_1 & & \end{array}$$

Some authors require that, for a base point $p \in B$, $\pi^{-1}(p) = F$, or that $\pi^{-1}(p) \cong F$ for every p. For a covering space and a manifold, the former and later respective requirement is a direct consequence of the respective definition: recall that a covering space is a surjective map $\pi : \widetilde{M} \longrightarrow M$ such that, for each $x \in M$, there exists a neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets, each of which is homeomorphic to U. The number of such disjoint open sets, called the evenly covered sets, is given by $|F|$. In this sense, a covering space is "locally discrete".

If F is a K-vector space of dimension n, then a fiber bundle is called a **vector bundle of rank** n, provided that the restriction $\varphi|_{\pi^{-1}(x)} : \pi^{-1}(x) \xrightarrow{\sim} \{x\} \times F = F_x$ is a K-linear map. A morphism between two vector bundles is, therefore, a morphism of two fiber bundles with the additional requirement that $\widetilde{f} : \pi_1^{-1}(x) \longrightarrow \pi_2^{-1} f(x)$ is a K-linear map.

Much like a manifold with an atlas, we need to consider what happens if, for an $x \in M$, we have overlapping, trivialising neighborhoods U and V of a vector bundle; so let $\varphi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times F$ and $\varphi_V : \pi^{-1}(V) \xrightarrow{\sim} V \times F$ be homeomorphisms. Observe that $\varphi_V \circ \varphi_U^{-1}$ is an endomorphism of $(U \cap V) \times F$ (modulo relevant domain restrictions). Since both φ_V and φ_U are isomorphisms, $\varphi_V \circ \varphi_U^{-1}$ is an isomorphism of the vector space $\{x\} \times F$ for each $x \in U \cap V$. To be more precise, since we are assuming that $U \cap V \neq \varnothing$ and, in particular, $x \in U \cap V$, the maps $\varphi_U|_{\pi^{-1}(x)}$ and $\varphi_V|_{\pi^{-1}(x)}$ are equal and well-defined isomorphisms of vector spaces. Let $g_{VU} : U \cap V \longrightarrow GL_n(K)$ denote the function that assigns this isomorphism to x. These functions are called **transition functions**. The inverse g_{VU}^{-1} is defined by the K-linear isomorphism $\left(\varphi_V \circ \varphi_U^{-1}\right)^{-1} = \varphi_U \circ \varphi_V^{-1}$, which is precisely g_{UV}. That is, $g_{VU}^{-1} = g_{UV}$. Now for $x \in U \cap V \cap W$, the composition of two such transition functions g_{UV} and g_{VW} is readily a K-linear isomorphism. However, observe that $\varphi_U \circ \varphi_W^{-1}$ is an endomorphism of $(U \cap V \cap W) \times F$ and that $\varphi_U \circ \varphi_W^{-1} = \varphi_U \circ id_{U \cap V \cap W} \circ \varphi_W^{-1} = \varphi_U \circ \left(\varphi_V^{-1} \circ \varphi_V\right) \circ \varphi_W^{-1} = \left(\varphi_U \circ \varphi_V^{-1}\right) \circ \left(\varphi_V \circ \varphi_W^{-1}\right)$. Thus, $g_{UW} = g_{UV} g_{VW}$, which is called the cocycle condition.

This additional information, of course, depends on the nature of F. Regardless of what space F is, the restriction $\varphi|_{\pi^{-1}(x)}$ provides the sense in which each $x \in M$ is said to parameterise the space F_x. Such is the situation if M is a (differentiable) manifold and E is its **tangent bundle** TM, in which case the fiber F_x is actually the tangent space $T_x M$. The tangent bundle is constructed as the disjoint union of tangent spaces $T_x M$ over $x \in M$. Thus, the tangent bundle is a special case of the vector bundle. Corresponding to the trivial fiber bundle, the trivial vector bundle is similarly given by $E = M \times F$ where F is a vector space. A **line bundle** is a vector bundle of rank 1. A special kind of vector bundle, called the **tensor bundle**, comprises of vector spaces V over K and their duals $V^\vee := \mathtt{Hom}_K(V, K)$. For example, if

$$^{(l,k)}(V) := \underbrace{V \otimes V \otimes ... \otimes V}_{l \text{ copies}} \otimes \underbrace{V^\vee \otimes V^\vee \otimes ... \otimes V^\vee}_{k \text{ copies}}$$

called **covariant tensors** on V of rank l and **contravariant tensors**[1] of rank k, then a **tensor bundle** may be defined as a disjoint union of (covariant and/or contravariant) tensors on tangent spaces $T_x M$ over $x \in M$. This machinery allows us to collect alternating contravariant tensors of rank k to define the vector space $\Lambda^k(V^\vee) \subset^{(0,k)} (V^\vee)$ of very special K-linear maps called **exterior forms**. In fact, the (vector) bundle $\Lambda^k(T^\vee M)$ is constructed as the disjoint union over $x \in M$ of $\Lambda^k(T_x^\vee M)$.

Bundles allow for a coordinate-free definition of fields: if $\pi : TM \longrightarrow M$ is a tangent bundle, then a **vector field** is a map $f : M \longrightarrow TM$ such that $\pi \circ f = id_M$. That is, a (covariant) vector field is a section of the vector bundle. In similar vain, and according to our terminology, a contravariant vector field would thus be a section of the bundle $T^\vee M \longrightarrow M$. Similarly, a **tensor field** is a section of a tensor bundle. A section of $\Lambda^k(T^\vee M)$ over M is a special vector field, called the **differential form of rank** k or a **differential** k

[1] In Physics, this convention is reversed.

-form, the collection of which is denoted by $\Omega^k(M)$. Differential forms allow integration on (orientable) manifolds.

In the absence of a sensible notion of orientation, the natural generalization of this is a **density**. If V is a vector space over K, a density is, in particular, a function $\mu \in \mathtt{Hom}_K(V \times ... \times V, K)$ such that, for any $T \in \mathtt{Hom}_K(V, V)$,

$$\mu(Tv_1, ..., Tv_n) = |\det T|\, \mu(v_1, ..., v_n)$$

The concept of a density is closely tied with orientation. Let $\boldsymbol{o} \in \mathfrak{o}(V) := \mathtt{Hom}_K(V \times ... \times V, K)$ be a function such that for any $T \in \mathtt{End}_K(V)$,

$$\boldsymbol{o}(Tv_1, ..., Tv_n) = sign(\det T)\, \boldsymbol{o}(v_1, ..., v_n).$$

Such a function which in addition satisfies $|\boldsymbol{o}(v_1, ..., v_n)| = 1$ for linearly independent vectors $v_1, ..., v_n$ is called an **orientation**.

Note that for $\omega \in \Lambda^n(V^\vee)$, the map $|\omega| : V \times V \times ... \times V \longrightarrow K$, defined by

$$|\omega|(v_1, ..., v_n) := |\omega(v_1, ..., v_n)|$$

(assuming that the field K has a valuation defined on it) is a density. If $\mathcal{D}(V)$ is the collection of densities on V, then $\mathcal{D}(V)$ is a 1 dimensional vector space spanned by $|\omega|$ for any nonzero $\omega \in \Lambda^n(V^\vee)$. This is because if $\{v_1, ..., v_n\}$ are a basis for V and for any $\alpha \in \mathcal{D}(V)$, we have equality of functions

$$\alpha = \left(\frac{\alpha(v_1, ..., v_n)}{|\omega|(v_1, ..., v_n)}\right)|\omega|$$

The uni-dimensionality of $\mathcal{D}(V)$ follows since the functions above agree on the basis $\{v_1, ..., v_n\}$. Note that the value $|\omega|(v_1, ..., v_n)$ is nonzero, for otherwise ω (or, equivalently, $|\omega|$) would be trivial.

Similarly, the obvious re-arrangement of the equation $\omega = \boldsymbol{o}|\omega|$ tells us that $\mathfrak{o}(V)$ is uni-dimensional, as well.

The **density bundle** $\mathcal{D}(M)$ is then a smooth (line) bundle over M, constructed as the disjoint union of $\mathcal{D}(T_xM)$ over $x \in M$. By the above construction, it is clear that $\mathcal{D}(M) = \Lambda^n(T^\vee M) \otimes \mathfrak{o}(M)$, where $\mathfrak{o}(M)$ is the **orientation line bundle** on M where, for each $x \in M$, the fiber is $\mathfrak{o}(T_xM)$. Thus, if M is oriented, then $\mathcal{D}(M) \cong \Lambda^n(T^\vee M)$ and we have our familiar calculus on the manifold. A **density**, in general, is a section of the density bundle. Therefore, for an n-dimensional manifold M, a density is a tensor field, which in local coordinates x^i at a point $x \in M$, may be written as $\alpha(x)\, dx^1 \wedge dx^2 \wedge ... \wedge dx^n$ for some real-valued function α.

The collection of densities is written as $\Omega^{|0|}(M)$. To see why this makes sense, define $\Omega^{|-q|}(M) := \Omega^{n-q}(M) \otimes \mathfrak{o}(M)$ as the collection of twisted $(n-q)$-forms. A density is then a twisted 0-form viz. an element of $\Omega^n(M) \otimes \mathfrak{o}(M)$. An alternative way to see this is sections of the tensor product of $\Lambda^q(TM) \otimes \Lambda^n(T^\vee M) \otimes \mathfrak{o}(M) \cong \Lambda^{n-q}(T^\vee M) \otimes \mathfrak{o}(M)$. This explains the function $\alpha(x) \in \Lambda^0(TM) = C^\infty(M)$. Thus, for example, an element of $\Omega^{|-1|}(M)$ may be of the form, in local coordinates x^i at a point $x \in M$,

$$\alpha^j(x)\frac{\partial}{\partial x^j} \otimes \left(dx^1 \wedge dx^2 \wedge ... \wedge dx^n\right)$$

for some $j \in \{1, ..., n\}$.

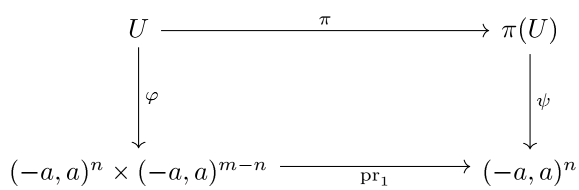
This allows us to define a graded module of twisted forms

$$\Omega^{|*|}(M) = \bigoplus_{q=0}^{n} \Omega^{|-q|}(M)$$

This is the de Rahm dg-algebra $\Omega^*(M)$ under wedge product but with the index reversed. That is, we have an exterior derivative $d : \Omega^{|-q|}(M) \longrightarrow \Omega^{|-q+1|}(M)$.

We need some more machinery. A **fibered bundle** (not 'fiber') is a bundle $\pi : E \longrightarrow M$ such that π is a submersion. That is, the pushforward $T\pi$ is also surjective. If E is a vector bundle, then surjectivity of $T\pi$ implies that the linear map $T_y\pi : T_yE \longrightarrow T_{\pi(y)}M$ has rank equal to the dimension of B. Therefore, every fiber bundle is a fibered bundle. However, for a fibered bundle, the fibers $\pi^{-1}(x)$ may not be the same space, let alone a vector space. For example, the natural projection map in the first variable $\pi : \mathbb{R}^2 \backslash \{(1,0)\} \longrightarrow \mathbb{R}$ is of course surjective and a submersion. However, $\pi^{-1}(1)$ is disconnected, unlike $\pi^{-1}(x \neq 1)$.

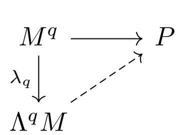
Recall that if $\dim E = m > \dim M = n$, then $T\pi$ is surjective. This is also useful in proving that the surjective map $\pi : E \longrightarrow M$ is a submersion if and only if for all $y \in E$, there exists a chart (U, φ) centered at y and a chart $(\pi(U), \psi)$ centered at $f(p)$ such that the following diagram commutes:



In general, surjective submersions are open maps. Thus, fibered bundles admit local sections $s \in \Gamma(U, E)$ for some open subset U of M. That is, for each $y \in E$, $\exists$ open subset $U \ni \pi(y)$ of M and a function $s : U \longrightarrow E$ such that $\pi \circ s = id_U$. Two local sections s_1 and s_2 about $x \in M$ are equivalent if, with respect to some adapted coordinate chart (and hence any adapted chart[2]), all the partial derivatives of s_1 and s_2 agree up to order k at x. This is clearly an equivalence relation, justifying the use of the symbol $j^k(s)(x)$ for the equivalence class of sections s at x whose partial derivatives agree up to order k. Let $J^k(E) = \{j^k(s)(x) : x \in M\}$ and define the jet bundle $\pi^{(k)} : J^k(E) \longrightarrow M$ with $\pi^{(k)}(j^k(s)(x)) = x$. It is customary to identify $J^0(E) = E$ and $\pi^0 = \pi$. If (U, φ) is a chart on E, then the k-**induced chart** (U^k, φ^k) on $J^k(E)$ is defined by $U^k = \{j^k(s)(x) : s(x) \in U\}$. This makes $J^k(E) \longrightarrow M$ a fibered manifold with the fiber over $x \in M$ being defined by $\{j^k(s)(x) : s \in \Gamma(M, E)\}$. Note that if sections s_1 and s_2 agree up to order j, then they agree up to all orders before i. Therefore, for $i \leqslant j$, we can define $\pi^i_j : J^j(E) \longrightarrow J^i(E)$ via $\pi^i_j(j^j(s)(x)) = j^i(s)(x)$. Note that, for $m \leqslant i \leqslant j$, $\pi^m_j = \pi^m_i \circ \pi^i_j$. This constitutes an inverse system and allows us to construct the infinite jet bundle $J^\infty(E)$ as an inverse limit in k of $J^k(E)$.

Background on Koszul Complexes

Let M be a module, R a ring and let M^q be the product of M with itself q times. We know that we can construct the module of alternating tensors $\Lambda^q M$ via



where P is any R-module. This allows us to define the skew-commutative graded algebra

$$\Lambda^* M = \bigoplus_n \Lambda^n M$$

with multiplication $\Lambda^k M \times \Lambda^l M \longrightarrow \Lambda^{k+l} M$ via $(a, b) \mapsto a \wedge b = (-1)^{ij}\, b \wedge a$. For $\omega : M \longrightarrow R$, the map

$$m_q : M^q \longrightarrow \Lambda^{q-1} M$$

defined via

$$(v_1, v_2, ..., v_q) \mapsto \sum_i \omega(v_i)(-1)^{i-1} v_1 \wedge v_2 \wedge ... \wedge \widehat{v_i} \wedge ... \wedge v_q$$

is easily seen to be multilinear and hence factors through

$$\omega_q : \Lambda^q M \longrightarrow \Lambda^{q-1} M$$

If $a \in \Lambda^k M$ and $b \in \Lambda^{q-k} M$, then $\omega_q(a \wedge b) = \omega_k(a) \wedge b + (-1)^k a \wedge \omega_{q-k}(b)$. To show this, consider the diagram

$$\begin{array}{ccc} M^k \times M^{q-k} & \longrightarrow & M^q \\ \downarrow & & \downarrow \\ \Lambda^k M \times \Lambda^{q-k} M & \longrightarrow & \Lambda^q M \end{array}$$

coming from two different constructions placed side-by-side. This diagram commutes because the upper arrow is a canonical isomorphism. The next thing to note is that

$$\omega_k(a) \wedge b + (-1)^k a \wedge \omega_{q-k}(b) = \omega_k(a) \wedge b \wedge \widehat{a} + (-1)^k a \wedge \omega_{q-k}(b) \wedge \widehat{b}$$

Couple this with $\omega_k \lambda_k = m_k$ and apply iteratively.

From this, it follows that $\omega_n \circ \omega_{n+1} = 0$ for all n and we have ourselves a complex, called the Koszul Complex of ω. To show this by (strong) induction, if we agree to let $\Lambda^{-1} M = \{0\}$, $\Lambda^0 M = R$, $\Lambda^1 M = M$ and $\omega_1 = \omega$, then the case for $n = 0$ is trivial. For $n = 1$,

$$\begin{aligned} \omega_1 \circ \omega_2 (a \wedge b) &= \omega_1(\omega_1(a) \wedge b - a \wedge \omega_1(b)) \\ &= \omega(\omega(a) b - \omega(b) a) \\ &= \omega(a)\omega(b) - \omega(b)\omega(a) = 0 \end{aligned}$$

Let $a \in \Lambda^k M$ and $b \in \Lambda^{q-1-k} M$ to give

$$\begin{aligned} (\omega_q \circ \omega_{q-1})(a \wedge b) &= \omega_q\left(\omega_k(a) \wedge b + (-1)^k a \wedge \omega_{q-1-k}(b)\right) \\ &= \omega_q(\omega_k(a) \wedge b) + (-1)^k \omega_q(a \wedge \omega_{q-1-k}(b)) \\ &= \omega_{k-1}(\omega_k(a)) \wedge b + (-1)^{k-1} \omega_k(a) \wedge \omega_{k-1}(b) + \\ &\quad (-1)^k \left[\omega_k(a) \wedge \omega_{q-k}(b) + (-1)^k a \wedge \omega_q \omega_{q-1-k}(b)\right] \\ &= (-1)^{k-1} \omega_k(a) \wedge \omega_{k-1}(b) + (-1)^k [\omega_k(a) \wedge \omega_{q-k}(b)] = 0 \end{aligned}$$

Lagrangian Field Theory

Consider the mapping space $\mathcal{F} = \Gamma(M, E) \subset \mathtt{Hom}_{Diff}(M, E)$ of sections of E over M. $\mathtt{Hom}_{Diff}(M, E)$ is a manifold in its own right, modelled after the Fréchet space, assuming that M and E are nice enough. Loosely speaking, M plays the role of the event space – the configuration space – and E plays the role of the universe of possible states. If E is a tangent bundle, then E is called the configuration manifold. If E is the cotangent bundle, then E is called the phase space. If $M = M^1$ is one-dimensional, then we can think of $\mathcal{F}$ as modelling particle's behavior over time on $E = \mathbb{R}^3$, say. Such is the case for Newtonian mechanics. A similar description holds for Minkowksi space $M = M^n$. In Hamilton's mechanics, the manifold M is replaced with $X \times M^1$ for some symplectic manifold X. The formalism of fields holds generally in that $\mathcal{F}$ plays the role of functions modelling a particle's behavior and, therefore, houses the phase space $\mathcal{M} \subset \mathcal{F}$. In physics parlance, an element of $\mathcal{F}$ is called a **field**. Thus, if $E = M \times \mathbb{R}$ is the trivial bundle, $\mathcal{F}$ comprises of scalar fields. For multiple fields ϕ_i considered simultaneously over bundles E_i, the fiber product $E = \bigtimes_i E_i$ may be utilized.

The goal of Lagrangian Field Theory is to determine functions $\phi \in \mathcal{F}$ that satisfy the Lagrangian operator used in conjunction with Principle of Least Action. That is, if $S \in \mathtt{Hom}_{Diff}(\mathcal{F}, \mathbb{R})$, then $\mathcal{M}$ is the critical manifold, (ideally) determined by functions satisfying $dS = 0$. One natural setting for this is in terms of differential forms on $\mathcal{F}$ and twisted forms on M. This allows us to model the Lagrangian density $L : \mathcal{F} \longrightarrow \mathcal{D}(M)$.

To see why, we first construct a double complex $\Omega^{\bullet,|\bullet|}(\mathcal{F} \times M)$ of fields $\mathcal{F}$ and twisted forms on M, with the (total) exterior derivative written as $D = \delta + d$, where $d : \Omega^{|-q|}(M) \longrightarrow \Omega^{|-q+1|}(M)$ and $\delta : \Omega^k(\mathcal{F}) \longrightarrow \Omega^{k+1}(\mathcal{F})$ obeying $d^2 = \delta^2 = 0$ and $d\delta = -\delta d$, so that $D^2 = 0$. For a fixed p, $\alpha \in \Omega^p(\mathcal{F})$ and $\beta \in \Omega^{|-q|}(M)$, we write $d(\alpha \wedge \beta) = (-1)^p \alpha \wedge d\beta$. One natural interaction of this double complex is captured in the following:

Lemma 1 *Let $\phi \in \mathcal{F} = \Gamma(M, E) \subset \mathtt{Hom}_{Diff}(M, E)$ and assume that M is compact. Then, $T_\phi \mathcal{F}$ is naturally isomorphic to the pullback vertical bundle $\Omega^0(M; \phi^*(E/M)) = \Gamma\left(\phi^*(E/M) \otimes \Lambda^0 T^\vee M\right) = \Gamma(M, \phi^*(E/M))$.*

Proof. First let $\mathrm{proj}_1 : E = M \times X \longrightarrow M$ be a trivial vector bundle. The general argument is similar. In this case, $\mathcal{F} = \mathtt{Hom}_{Diff}(M, E) = C^\infty(M, E)$. We must first construct the tangent bundle of E. One component of this is the tangent bundle $\pi_M : TM \longrightarrow M$. Moreover, since X is a vector space, $T_x X \cong X \times \{x\}$ (so that $TX = X \times X$). Thus, for $e = (m, x) \in M \times X = E$, we can have the tangent space $T_e E \cong T_m M \times T_x X$. We, therefore, have the following diagram:

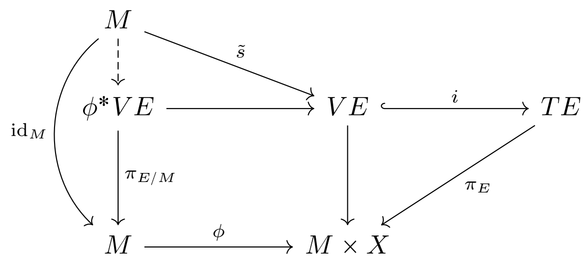
$$\begin{array}{ccccccccc} & & & & E = M \times X & \xrightarrow{\mathrm{proj}_1} & M & & \\ & & & & \uparrow{\scriptstyle \pi_E} & & \uparrow{\scriptstyle \pi_M} & & \\ 0 & \longrightarrow & \ker(T\mathrm{proj}_1) & \longrightarrow & TE & \xrightarrow{T\mathrm{proj}_1} & TM & \longrightarrow & 0 \end{array}$$

where

$$TM = \bigsqcup_{m \in M} T_m M,\ TE = \bigsqcup_{e \in E} T_e E \cong \bigsqcup_{(m,x) \in E} T_m M \times T_x X = TM \times TX$$

and, for each $e = (m, x)$, $\ker_e(T\mathrm{proj}_1) = \{m\} \times X$. Therefore, $\ker(T\mathrm{proj}_1) = M \times X \times X = M \times TX$ is the vertical tangent bundle $E/M = VE$ of E.

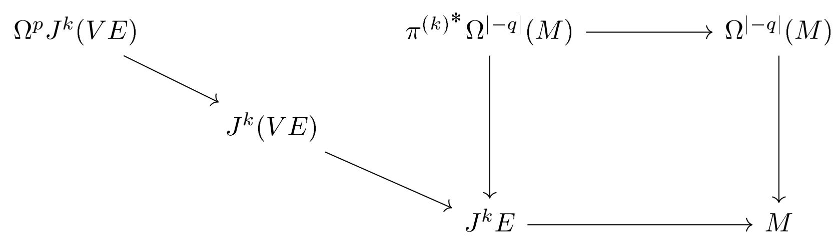
Now let ϕ be a section of E over M and consider the pullback bundle $\pi_{E/M} : \phi^*(E/M) \longrightarrow M$ and the diagram



The sections of this bundle – a dotted arrow above after a choice of $\tilde{s}$ – form the collection $\Omega^0\left(M; \phi^*\left(E/M\right)\right)$. The pullback bundle is constructed via the usual $\phi^*\left(E/M\right) = \{(m, \xi) \in M \times VE : \phi\left(m\right) = \left(\pi_E \circ i\right)\left(\xi\right)\}$. Now, $\mathcal{F}$ comprises of paths ϕ_ε defined $\varepsilon : I \longrightarrow \phi_\varepsilon$ over a real interval I containing 0 with $\phi_0 = \phi$ such that tangent vectors $s \in T_\phi \mathcal{F}$ satisfy the expected $s = \frac{d}{d\varepsilon}\phi_\varepsilon\big|_{\varepsilon=0}$. The second ingredient we need is the observation that $s \in T_\phi\mathcal{F} \iff s\left(m\right) \in T_{\phi(m)}E \subset VE$. This gives us our $\widetilde{s}$ and hence establishes the correspondence. ■

Lagrangian densities $L : \mathcal{F} \longrightarrow \mathcal{D}\left(M\right)$ are elements of $\Omega^{0,|0|}\left(\mathcal{F} \times M\right)$, where $L\left(\phi\right)$ is a density. A zero-form on $\mathcal{F}$ is a smooth function and gets absorbed in the density. Lagrangians are local. Loosely, this means one considers energy distribution around the 'neighborhood' of a particle whose Lagrangian is under consideration.

This can be made mathematically precise. Consider the form $\alpha \in \Omega^{p,|-q|}\left(\mathcal{F} \times M\right)$ at a point $\left(\phi, m\right) \in \mathcal{F} \times M$. If $\xi_1, ..., \xi_p \in T_\phi\mathcal{F}$, the twisted $(n-q)$-form $\alpha_{(\phi,m)}\left(\xi_1, ..., \xi_p\right)$ at m is said to be **local** if, for some integer k, α only depends on $j^k\left(\phi\right)\left(m\right)$ and $j^k\left(\xi_i\right)\left(m\right)$ for $1 \leqslant i \leqslant p$. This definition can be recast in terms of sections, and requires the following ingredients: by **Lemma 1**, each ξ_i corresponds to a section of $\phi^*\left(E/M\right) = \phi^*\left(VE\right)$; the k-jet bundle $\pi^{(k)} : J^kE \longrightarrow M$ gives us p-forms $\Omega^p\left(J^kE/M\right)$, and, of course, we need to pull back the bundle $\Omega^{|-q|}\left(M\right) \longrightarrow M$ through $\pi^{(k)}$:



Together, these bundles over J^kE give us the bundle $\sigma : \Omega^p\left(J^kE/M\right) \otimes \pi^{(k)^*}\Omega^{|-q|}\left(M\right) \longrightarrow J^k\left(E\right)$. A section of this bundle is a local form. In particular, local forms depend on vector fields on M.

The collection of such local forms gives us a subcomplex $\left(\Omega^{p,|*|}_{loc}\left(\mathcal{F} \times M\right), d\right)$ with differential

$$d : \Omega^p\left(J^kE/M\right) \otimes \pi^{(k)^*}\Omega^{|-q|}\left(M\right) \longrightarrow \Omega^p\left(J^{k+1}E/M\right) \otimes \pi^{(k+1)^*}\Omega^{|-q+1|}\left(M\right)$$

Takens' Theorem

The main theorem, originally presented in [6], is the following:

Theorem 2 (Takens) *For $p > 0$, the complex $\left(\Omega^{p,|\bullet|}_{loc}\left(\mathcal{F} \times M\right), d\right)$ of local (twisted) forms is exact, except in the top degree $|\bullet| = 0$*

This follows directly from the following generalization:

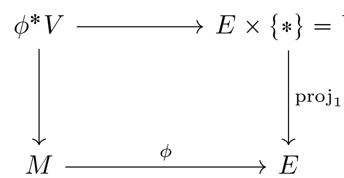
Theorem 3 (Takens) *Let* $E \longrightarrow M$ *be a submersion,* $p \in \mathbb{Z}_{>0}$ *and let* $V_i \longrightarrow E$ *be vector bundles for* $i = 1, ..., p$*. If*

$$V = \prod_E V_i$$

is the fiber product over E*,* $\phi \in \Gamma(M, E)$ *is a section, and if we let* $\mathcal{V}_\phi$ *be the space of sections of* $\phi^* V \longrightarrow M$*, then the subcomplex* $\left(\Omega^{0,\bullet}_{loc,mult}(\mathcal{V}_\phi \times M), d\right)$ *of forms* $\alpha(\phi, \xi_1, ..., \xi_p)$ *which are* $\mathbb{R}$*-multilinear in* ξ_i*, is exact, except in the top degree* $\bullet = 0$*.*

The forms $\alpha(\phi, \xi_1, ..., \xi_p)$ depend locally on ϕ and on sections ξ_i of $\phi^* V_i$.

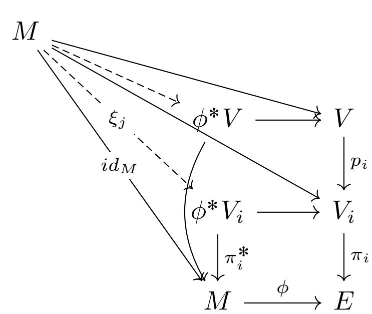
The former follows from latter for, say $p = 1$, if we agree to call this case that of the trivial vector bundle $E \times \{*\} \longrightarrow E$. We have $\mathcal{V}_\phi = \Gamma(M, E)$ by virtue of commutativity of the diagram



since $\phi^* V = \{(m, e_0) \in M \times V : \phi(m) = \text{proj}_1(e, *) = e\}$, which is just the graph of ϕ and, therefore, corresponds to the image of ϕ.

We want $\Omega^{0,\bullet}_{loc,mult}(\mathcal{V}_\phi \times M)$ to correspond to $\Omega^{p,|\bullet|}_{loc}(\mathcal{F} \times M)$. What we have at our disposal is a fibration $V \xrightarrow{\Pi} E$ from $V \xrightarrow{\text{proj}_i} V_i \xrightarrow{\pi_i} E$ with $\pi_i \circ \text{proj}_i = \pi_j \circ \text{proj}_j$ for all $i, j \in \{1, ..., p\}$. We assume that E is connected, in which case all fibers V_i are isomorphic. Thus, V is a vector bundle over E with fibers isomorphic to V_i. We want to consider bundles over M, so it is natural to invoke pullbacks, in which case $\phi^* V = \{(m, v) \in M \times V : \phi(m) = \Pi(v)\}$. The form $\alpha \in \Omega^{0,\bullet}_{loc,mult}(\mathcal{V}_\phi \times M)$ is local when it depends on the k-jet of the section ϕ of E/M and the k-jet of sections ξ_i of $\phi^* V_i$. To see this, first observe that $\Gamma_\phi(M, V) \cong \mathcal{V}_\phi$, thus justifying the replacement for $\mathcal{F}$ and the choice of section ϕ. The isomorphism holds because each unlabelled dashed arrow in the diagram below is an element of $\mathcal{V}_\phi$. The latter part of the definition is justified as a consequence of **Lemma 1**. In fact, if all of the vector bundles are replaced with the vertical bundle, then the proof of **Lemma 1** tells us that $\Omega^{p,|\bullet|}_{loc}(\mathcal{F} \times M)$ is the antisymmetric part of $\Omega^{0,\bullet}_{loc,mult}(\mathcal{V}_\phi \times M)$. Finally, the bundle $\phi^* V \longrightarrow M$ can be given a consistent orientation, so that twisted forms are isomorphic to ordinary forms.

Note that the sections ξ_i correspond to $f_i, g_i : M \longrightarrow V_i$ and that $\pi_i \circ f_i = \pi_i \circ g_i = \phi$ because $\phi^* V_i$ is the categorical pull-back. The pull-back lemma implies that the complete diagram commutes:



The proof is found in [3], pp 188-190.

Proof. Let $\pi : E \longrightarrow M$ be a submersion with M of rank m and let $\pi(e_0) = m_0$. Let (U, x) be chart containing m_0 such that $W = F \times U \cong \pi^{-1}(U)$ with F a rank f vector space. After appropriate local trivializations of V_i on W, we can write, say, $\alpha \in \Omega^{0,|-q|}_{loc,mult}\left(\mathcal{V}_\phi \times M\right)$ as

$$\alpha\left(\phi, \xi_1, ..., \xi_p\right) = \sum \alpha_{n_1,...,n_p}\left(\phi\right) \partial^{n_1}\xi_1 ... \partial^{n_p}\xi_p \tag{1}$$

The indices n_i would have been simpler, if we had $\mathcal{V}_\phi = \mathcal{F}$. However, since sections are of $\phi^* V$ over M and V is a fiber product of V_i (over E), each $n_i = \left(n_{i_1}, n_{i_2}, ..., n_{i_m}\right)$ is a multi-index and the operator $\partial^{n_i} := \partial_1^{n_{i_1}} \partial_2^{n_{i_2}} ... \partial_m^{n_{i_m}}$ in a chosen coordinate system $x = \left(x^1, ..., x^m\right)$. The order of ∂^{n_i} is given by

$$|n_i| = \sum_{j=1}^{m} n_{i_j}.$$

Since α is local, by definition, for some $k \in \mathbb{Z}$, each $\alpha_{n_1,...,n_p}$ depends only on the k-jet ϕ and so, we can write

$$\alpha_{n_1,...,n_p} : J^k\left(E/M\right) \longrightarrow \pi^{(k)^*}\left(\Omega^q\left(M\right)\right) \otimes \bigotimes_{i=1}^{p} \pi^{(k)^*} V_i^\vee$$

where $\pi^{(k)} : J^k\left(E/M\right) \longrightarrow M$. In order to define a complex, we need to have an increasing index, which we can sort by order of the (partial) derivative. One problem, however, is that the order of ∂^{n_i} is agnostic about the permutation of the indices n_{i_j}. If we let

$$N = \sum_{i=1}^{p} |n_i|,$$

where $|n_i| = n_{i_1} + n_{i_2} + ... + n_{i_m}$, we can construct an increasing filtration, for each q,

$$F_0 \subseteq F_1 \subseteq ... \subseteq F_N \subseteq F_{N+1} \subseteq ...$$

with the differential $d_N : F_N \longrightarrow F_{N+1}$. In local coordinates, the differential would be

$$d\left(\sum \alpha_{n_1,...,n_p}\left(\phi\right) \partial^{n_1}\xi_1 ... \partial^{n_p}\xi_p\right) = \sum \left(-1\right)^p \alpha_{n_1,...,n_p}\left(\phi\right) d\left(\partial^{n_1}\xi_1 ... \partial^{n_p}\xi_p\right)$$

Note that because the grading is based on order of the derivative, and the derivative ∂^{n_i} is independent of the choice of local coordinates. Let $\mathcal{J}_k$ be the collection of sections α below:

$$\alpha : J^k\left(E/M\right) \longrightarrow \bigotimes_{i=1}^{p} \bigotimes_{|n_i| \leqslant k} \pi^{(k)^*} Sym^{|n_i|}\left(TM\right) \otimes \pi^{(k)^*} V_i^\vee \otimes \pi^{(k)^*}\left(\Omega^q\left(M\right)\right) \tag{2}$$

We construct $Gr^F\left(\Omega^{0,q}_{loc,mult}\right)$ as a limit of the following cone

$$\begin{array}{ccc} \mathcal{J}_i & \xrightarrow{f_{ij}} & \mathcal{J}_j \\ & \searrow \quad \swarrow & \\ & Gr^F_q & \end{array}$$

where, for $i \leqslant j$, the transition f_{ij} map is the pullback of $\pi^i_j : J^j\left(E/M\right) \longrightarrow J^i\left(E/M\right)$ via $\pi^i_j\left(j^j\left(s\right)\left(x\right)\right) = j^i\left(s\right)\left(x\right)$. The idea here is that each (coordinate free) derivative ∂^{n_i} has a corresponding symbol in

$Sym^{|n_i|}(TM)$ and the degree N is the sum of the degrees $|n_i|$. This allows us to write

$$Gr^F\left(\Omega^{0,q}_{loc,mult}\right) = \bigoplus_{N=0} F_{N+1}/F_N := \bigoplus_{N=0} Gr^q_N$$

In fact, the differential d_N induces

$$d : Gr^q_N \longrightarrow Gr^{q+1}_{N+1}$$

via **(2)** as

$$\begin{aligned} d \quad : \quad & \bigotimes_{i=1}^{p} \bigotimes_{|n_i|\leqslant k} \pi^{(k)^*} Sym^{|n_i|}(TM) \otimes \pi^{(k)^*} V_i^{\vee} \otimes \pi^{(k)^*}(\Omega^q(M)) \\ \longrightarrow \quad & \bigotimes_{i=1}^{p} \bigotimes_{|n_i|\leqslant k+1} \pi^{(k+1)^*} Sym^{|n_i|}(TM) \otimes \pi^{(k+1)^*} V_i^{\vee} \otimes \pi^{(k+1)^*}\left(\Omega^{q+1}(M)\right) \end{aligned}$$

Modulo tensorization with $V_i^{\vee}$ and the pullback, d can be described by the map

$$\bigotimes_1^p Sym^*(TM) \otimes \Omega^q(M) \longrightarrow \bigotimes_1^p Sym^*(TM) \otimes \Omega^{q+1}(M)$$

in local terms: if $\{e_l\}$ is a basis for TM, $s \in \bigotimes_{j=1}^{p} Sym^*(TM)$ and $\zeta \in \Omega^q(M)$, then

$$s \otimes \zeta \mapsto \left(\sum_l \left(\sum_{i=1}^{p} 1 \otimes ... \otimes (e_l \text{ at the } i\text{th place}) \otimes ... \otimes 1\right).s\right) \otimes e^l \wedge \zeta \tag{3}$$

We can, therefore, consider the complex

$$\text{part of degree } N+q \text{ of } \bigotimes_1^p Sym^*(TM),$$

tensored with $\Omega^q(M)$. The chain map for this complex is given by Eq **(3)**. We will be done if we can show that this complex is exact, except in the top degree. We do this point-by-point, so let $m_0 \in M$. Our focus then turns to a complex with components

$$\bigotimes_1^p Sym^*(T_{m_0}M) \otimes \Lambda^q\left(T^{\vee}_{m_0}M\right)$$

Recall that

$$\bigotimes_1^p Sym^*(T_{m_0}M) = Sym^*\left(\bigoplus_1^p T_{m_0}M\right)$$

The homeomorphism $\Delta : T_{m_0}M \longrightarrow T_{m_0}M \times T_{m_0}M \times ... \times T_{m_0}M$ (p-times) given by $t \mapsto (t, t, ..., t)$ allows us to identify $T_{m_0}M$ within $\bigoplus_1^p T_{m_0}M$. Thus, we can write, for some subspace S,

$$\bigoplus_{j=1}^{p} T_{m_0}M = S \oplus \Delta(T_{m_0}M) \cong S \oplus T_{m_0}M$$

which allows us to re-write

$$Sym^{*}\left(\bigoplus_{1}^{p}T_{m_0}M\right)=Sym^{*}\left(S\oplus T_{m_0}M\right)=Sym^{*}\left(S\right)\otimes Sym^{*}\left(T_{m_0}M\right)$$

Thus, the complex locally is

$$\bigotimes_{1}^{p}Sym^{*}\left(T_{m_0}M\right)\otimes\Lambda^{q}\left(T_{m_0}^{\vee}M\right)=Sym^{*}\left(S\right)\otimes Sym^{*}\left(T_{m_0}M\right)\otimes\Lambda^{q}\left(T_{m_0}^{\vee}M\right)$$

The homological degree of $Sym^{*}\left(S\right)$ is zero and the degree of latter two combined is q. The complex is then $\left(Sym^{*}\left(T_{m_0}M\right)\otimes\Lambda^{q}\left(T_{m_0}^{\vee}M\right),d\right)$ where d is again induced by Eq **(3)**: if $\{e_l\}$ are the basis for $T_{m_0}M$ and $\left\{e^l\right\}$ are the dual basis for $T_{m_0}^{\vee}M$, then

$$d:Sym^{*}\left(T_{m_0}M\right)\otimes\Lambda^{q}\left(T_{m_0}^{\vee}M\right)\longrightarrow Sym^{*}\left(T_{m_0}M\right)\otimes\Lambda^{q+1}\left(T_{m_0}^{\vee}M\right)$$

given by

$$s\otimes\zeta\mapsto\sum_{l}\left(\left(e_l.s\right)\otimes\left(e^l\wedge\zeta\right)\right)$$

Note that if we write $T_{m_0}M=A\oplus B$, then

$$\begin{aligned}Sym^{*}\left(A\oplus B\right)\otimes\Lambda^{\bullet}\left(A^{\vee}\oplus B^{\vee}\right)&\cong Sym^{*}\left(A\right)\otimes Sym^{*}\left(B\right)\otimes\left(\Lambda^{\bullet}\left(A^{\vee}\right)\otimes\Lambda^{\bullet}\left(B^{\vee}\right)\right)\\&\cong\left(Sym^{*}\left(A\right)\otimes\Lambda^{\bullet}\left(A^{\vee}\right)\right)\otimes\left(Sym^{*}\left(B\right)\otimes\Lambda^{\bullet}\left(B^{\vee}\right)\right)\end{aligned}$$

Breaking up the finite dimensional space $T_{m_0}M$ into one-dimensional components gives us one-dimensional respective complexes. Moreover, the equivalence

$$\Lambda^{q}\left(A^{\vee}\oplus B^{\vee}\right)\cong\bigoplus_{a+b=q}\Lambda^{a}A^{\vee}\otimes\Lambda^{b}B^{\vee}$$

tells us that to check acyclicity at the top degree, we might as well focus on the case when $\dim T_{m_0}M=1$. Thus, the case we have is that of

$$\begin{aligned}Sym^{*}\left(K\right)\otimes\Lambda^{\bullet}\left(K^{\vee}\right)&\cong K\left[X\right]\otimes\left(K\oplus K^{\vee}\right)\\&\cong\left(K\left[X\right]\otimes K\right)\oplus\left(K\left[X\right]\otimes K^{\vee}\right)\end{aligned}$$

for a field K of characteristic zero and the only non-trivial differential m_X we have is

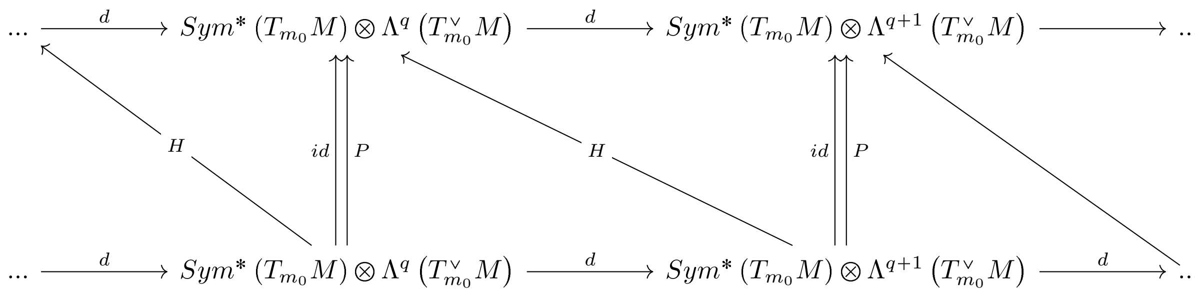
$$0\longrightarrow K\left[X\right]\cong K\left[X\right]\otimes K\xrightarrow{m_X}K\left[X\right]\otimes K^{\vee}\cong K\left[X\right]\longrightarrow 0$$

If γ is some scalar, then $f\left(X\right)\otimes\gamma=\left(\gamma f\right)\left(X\right)\otimes 1\longleftrightarrow\left(\gamma f\right)\left(X\right)=g\left(X\right)$ gets mapped to $d\left(g\left(X\right)\right)=Xg\left(X\right)\otimes dX=Xg\left(X\right)\otimes 1\longleftrightarrow Xg\left(X\right)$ and so, m_X is multiplication by X. The homology in degree 0 is $H_0=\ker m_X/\{0\}\cong\{0\}$ whereas homology in degree 1 is $H_1=\mathbb{R}\left[X\right]/m_X\left(\mathbb{R}\left[X\right]\right)\cong\mathbb{R}$.

Another way to show the same argument is as follows: define homotopy H as

$$H\left(s\otimes\zeta\right)=\begin{cases}0 & \text{if }\deg s=0\text{ and if }\deg\zeta=\dim T_{m_0}M\\ \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l \partial_{e^l}\left(s\right)\otimes\iota_{e_l}\left(\zeta\right) & \text{otherwise}\end{cases}$$

and extend linearly, where $\iota_{e_l}: T^{\vee}_{m_0}M\longrightarrow K$ is defined by $\iota_{e_l}\left(\zeta\right)\left(\rho_1,...,\rho_{q-1}\right)=\zeta\left(e_l,\rho_1,...,\rho_{q-1}\right)$. If $P: Sym^*\left(T_{m_0}M\right)\otimes\Lambda^q\left(T^{\vee}_{m_0}M\right)\longrightarrow Sym^0\left(T_{m_0}M\right)\otimes\Lambda^n\left(T^{\vee}_{m_0}M\right)\subset Sym^*\left(T_{m_0}M\right)\otimes\Lambda^q\left(T^{\vee}_{m_0}M\right)$ where $\dim T_{m_0}M=n$ and P defined using the natural maps available.



Now, if $\deg s=0$ and if $\deg\zeta=\dim T_{m_0}M$, then d is the trivial map and so is $dH+Hd$ and the identity map coincides with P. In the other case,

$$\begin{aligned}
\left(dH+Hd\right)\left(s\otimes\zeta\right) &= dH\left(s\otimes\zeta\right)+Hd\left(s\otimes\zeta\right)\\
&= d\left(\frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l\partial_{e^l}\left(s\right)\otimes\iota_{e_l}\left(\zeta\right)\right)+H\left(\sum_l\left(\left(e_l.s\right)\otimes\left(e^l\wedge\zeta\right)\right)\right)\\
&= \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l d\left(\partial_{e^l}\left(s\right)\otimes\iota_{e_l}\left(\zeta\right)\right)+\sum_l H\left(\left(e_l.s\right)\otimes\left(e^l\wedge\zeta\right)\right)\\
&= \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l\sum_j\left(e_j.\partial_{e^l}\left(s\right)\right)\otimes\left(e^j\wedge\iota_{e_l}\left(\zeta\right)\right)+\\
&\quad\sum_l\frac{1}{\deg e_l.s+\dim T_{m_0}M-\deg e^l\wedge\zeta}\sum_j\partial_{e^j}\left(e_l.s\right)\otimes\iota_{ej}\left(e^l\wedge\zeta\right)\\
&= \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l\sum_j\left(e_j.\partial_{e^l}\left(s\right)\right)\otimes\delta_{jl}\zeta+\\
&\quad\sum_l\frac{1}{\deg e_l.s+\dim T_{m_0}M-\deg e^l\wedge\zeta}\sum_j\partial_{e^j}\left(e_l.s\right)\otimes\delta_{lj}\wedge\zeta\\
&= \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}\sum_l\left(e_l.\partial_{e^l}\left(s\right)\right)\otimes\zeta+\\
&\quad\sum_l\frac{1}{\deg e_l.s+\dim T_{m_0}M-\deg e^l\wedge\zeta}\partial_{e^l}\left(e_l.s\right)\otimes\zeta\\
&= \frac{1}{\deg s+\dim T_{m_0}M-\deg\zeta}s\otimes\zeta+\\
&\quad\sum_l\frac{1}{\deg e_l.s+\dim T_{m_0}M-\deg e^l\wedge\zeta}\left(s+\left(-1\right)^{|s|}e_l\partial_{e^l}\left(s\right)\right)\otimes\zeta\\
&= s\otimes\zeta-P\left(s\otimes\zeta\right)
\end{aligned}$$

Thus, $dH+Hd=id-P$.

Now, to show that this local argument can be extended globally, let $\left\{W_j\right\}_{j\in I}$ be a cover of E and let

$\{\chi_j\}_{j\in I}$ be a partition of unity $\{W_j\}_{j\in I}$ such that the closure $\overline{\{e:\chi_j(e)\neq 0\}}\subset W_j$. That is, $\{\chi_j\}$ is a collection of maps of compact support on E to $[0,1]$ such that for every $e\in E$, (a) $\exists$ a neighborhood $V\subset E$ of e such that all but finitely many χ_j are zero and (b)

$$\sum_j \chi_j(e) = 1$$

We can write a corresponding equation to **(1)** in local coordinates on each W_j, end up with homotopy H_j as above and then define homotopy $\mathcal{H}(\alpha) = \Sigma H_j(\chi_j(\alpha))$, we again have

$$id-(dH_j+H_jd) = P : Sym^*(S)\otimes Sym^*(T_{m_0}M)\otimes\Lambda^q\left(T^\vee_{m_0}M\right) \longrightarrow Sym^*(S)\otimes Sym^0(T_{m_0}M)\otimes\Lambda^n\left(T^\vee_{m_0}M\right)$$

glued together to give $id-(d\mathcal{H}+\mathcal{H}d)$. That is, to yield a map $F_N\longrightarrow F_{N-1}$. If $\alpha\in F_N$ such that $d\alpha=0$, then

$$\alpha' := (id-(dH+Hd))(\alpha) = id(\alpha)-(dH+Hd)(\alpha) = \alpha - dH(\alpha) + Hd(\alpha) = \alpha - dH(\alpha)$$

is in F_{N-1}. Thus, $\alpha = \alpha' + dH(\alpha)$. Note that $d(\alpha') = d(\alpha - dH(\alpha)) = d\alpha - d(dH(\alpha)) = 0$ and so, α' is closed. Thus, we can repeat the above argument to have α'' closed such that $\alpha' = \alpha'' + dH(\alpha')$, eventually ending up with $\alpha^{(N)}\in F_{-1}$ and $\alpha = \alpha^{(N)} + d\,(\text{terms}) = d\beta$. ■

In Classical Field Theory, the k-jets for Lagrangian Densities L (generally called **source forms**) are usually restricted to the case of $k=1$ and so, it defines a $(1,|-1|)$-form, called a variational 1-form γ with $D\mathcal{L} = \delta L + d\gamma$. By Takens' Theorem, this γ is guaranteed to exist. The decomposition of DL is unique, up to $d\beta$ where β is a local $(1,|-1|)$-form. The Euler Lagrange Equations, following the principle of least action, are given by $DL=0$, which cut out phase space $\mathcal{M}$ of fields, via differential equations of a system under investigation. Moreover, γ can also be used to determine a conservation law of the Lagrangian system under consideration. This is the content of [7]. In fact, the pair (L,γ) *defines* a field theory.

Future Work

We have seen that jet bundles describe the way local forms in general and Lagrangians in particular operate, answering questions about existence of Lagrangians and their variations. However, there are cases where the action functional may not exist and thus Principle of Least Action fails to be defined, even if, as is always the case, the variation is always defined. Dissipative forces are a guiding example, and a workaround for them might be in extension of the phase space or considering multiple interacting systems on the same manifold. This leads to multiple Lagrangians in different coordinate frames, which raises issues of gluing the Lagrangians. A possible answer is via multivalued jets[1] for Classical Field Theory. A natural step further is to use Lie algebroids as a generalization of the jet bundles. This might offer a way to strengthen the variational complex for general field theories. One challenge that may arise is that gluing of these 'higher jet bundles' may not behave very nicely. To aid this development, Lie algebroids can be treated as L_∞ spaces[4], allowing ready translation to the gluing problem. Our future line of work is then an investigation into the translation of the infinite jet bundle as a Lie algebroid first and, later on, applying it to a variational bicomplex for general field theories.

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